What is a theory according to the anthropological theory of the didactic?

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The question tackled here centres on the notion—or, more precisely, the many notions—of theory often used in discussing scientific matters. The analysis that we attempt develops within the framework of the anthropological theory of the didactic (ATD). It purports to show that current usage refers mostly to the "emerged parts" of so-called theories and largely ignores their "immersed parts", which are the correlate of their intrinsic implicitness and historical incompleteness. This leads to favour open theorization over entrenched theory.

Keywords: Theorization, praxeology, knowledge, human activity, institutions.

INTRODUCING THE NOTION OF THEORY

In this study we examine the meaning and scope of a key concept of ATD which, paradoxically, since the inception of this theory, seems to have been consistently overlooked: that of *theory*. A word akin to English "theory" exists in many European languages [1]. According to John Ayto's *Dictionary of Words Origins* (1990), the history of *theory* goes as follows:

> **theory** [16] The etymological notion underlying *theory* is of 'looking'; only secondarily did it develop via 'contemplation' to 'mental conception.' It comes via late Latin *theōria* from Greek *theōríā* 'contemplation, speculation, theory.' This was a derivative of *theōrós* 'spectator,' which was formed from the base *thea-* (source also of *theā́sthai* 'watch, look at,' from which English gets *theatre*). Also derived from *theōrós* was *theōreĩn* 'look at,' which formed the basis of *theṓrēma* 'speculation, intuition, theory,' acquired by English via late Latin *theōrēma* as *theorem* [16]. From the same source comes *theoretical* [17]. (p. 527)

A paper by a classical scholar, Ian Rutherford, gives more information on the uses of the word *theoria* in Ancient Greece:

> The Greek word *theoria* means "watching," and has two special senses in Greek culture: first, a religious delegation sent by a Greek city, to consult an oracle or take part in a festival at a sanctuary outside its territory, and second, philosophical contemplation. *Theoria* in the first sense is attested from the sixth century bce until the Roman Empire, but the sources are particularly rich in the Hellenistic period. Sacred delegates were called *theoroi*, were often led by a so-called *architheoros*, and if they went by sea, the vehicle was a *theoris*-ship. (Abstract)

The first of these two senses has almost disappeared from modern usage. The second sense opened the way for our common uses of *theory*. In the following, we concentrate on "modern" meanings of this word, which dictionaries usually condense into a small number of categories, as does for example the English *Wiktionary*. The entry dedicated to *theory* in this dictionary begins classically with the etymology of the word, then passes on to the uses of it that it does retain:

> **theory** (*countable and uncountable*, *plural* theories)
>
> 1) (obsolete) Mental conception; reflection, consideration. [16th-18th c.]
>
> 2) (sciences) A coherent statement or set of ideas that explains observed facts or phenomena, or which sets out the laws and principles of something known or observed; a hypothesis confirmed by observation, experiment etc. [from 17th c.]

3) (uncountable) The underlying principles or methods of a given technical skill, art etc., as opposed to its practice. [from 17th c.]

4) (mathematics) A field of study attempting to exhaustively describe a particular class of constructs. [from 18th c.] *Knot theory classifies the mappings of a circle into 3-space.*

5) A hypothesis or conjecture. [from 18th c.]

6) (countable, logic) A set of axioms together with all statements derivable from them. *Equivalently*, a formal language plus a set of axioms (from which can then be derived theorems). *A theory is consistent if it has a model.*

In what follows we shall draw upon such semantic summaries in order to suggest that the notion of theory developed in ATD can account for the diversity of usages that exist today.

SOME BASICS OF ATD

In ATD, the basic "entities" are *persons x* and *institutions I*. These notions are close to their ordinary counterparts, although they are more general: in ATD, a newborn infant is a person; and, to take just one easy example, a class, with its students and teachers, is an institution. An institution *I* comprises different *positions p*— in the case of a class, that of student and of teacher. To every person *x* or institutional position *p* is assigned a "praxeological equipment", which is the system of "capacities" that, under appropriate conditions, enables the person *x* or any person *x'* occupying position *p* to act and think through one's actions.

Any praxeological equipment, be it *personal* or *positional*, is made up of, among other things, "notions". Most persons and institutional positions thus have a certain notion of theory—if only through the overused phrase "in theory". The present study could then be said to be partly about the notion of theory in ATD (taken as an institution). However that may be, it is essential to detach oneself from the seemingly undisputed belief that there would exist a unique, shared *notion* of theory of which the meaning would simply vary according to the context of use. In ATD, every person, every institutional position is supposed to be endowed with a peculiar notion of theory, that notion being shaped by the *constraints* to which the person or position is currently subjected. This phenomenon is at the origin of the processes of institutional transposition, of which didactic transposition is but a particular case (Chevallard, 1992). In order to make headway, we shall now delineate the "anthropological" notion of theory—which, at the start, is only one such notion amongst others.

THE NOTION OF PRAXEOLOGY IN ATD

ATD posits a theory of human activity that hinges on an essential and founding notion: that of *praxeology* (Chevallard, 2006, 2015; see also Bosch & Gascón, 2014). The word *praxeology* has been around for (at least) two centuries in the sense recorded by most dictionaries, in which it is held to refer to the "study of human action and conduct", to the "study of practical or efficient activity", or to the "science of efficient action". The use made here of the word pertains properly to ATD and departs decisively from this old-established, though infrequent, use. A key tenet of ATD is that when a person *x* acts purposely and knowingly, her doings can be analysed into a (finite) sequence of *tasks* t_1, t_2, ..., t_n. Contrary to the common meaning of the term (which has a ring of unpleasantness about it), *task* is taken here in a very general sense, irrespective of its volume or pettiness: *to open this door* and *to smile to this neighbour* are tasks; *to scratch this person's back*, *to write this sonnet*, *to save this polar bear*, *to prove this theorem*, and *to play this guitar chord* are tasks as well. Any task *t* is regarded as a "specimen" of a *type* of tasks *T*. In order to execute the task *t* of type *T*, a person *x* draws on a determined *technique*, denoted τ_T, that is to say a (more or less precise) way of accomplishing (at least some) tasks *t* of type *T*. No technique τ can cope with the totality of tasks of a given type *T*—its range of success is usually called the *scope* of τ. If, for example, it is clear that elementary techniques for factoring numbers all have a limited scope, it is true also, for obvious reasons, that *any* technique whatsoever eventually reaches its limits.

Let us take another example, that of a technique for finding the quotient of number *a* by number *b* (with $a, b \in \mathbb{N}^*$), which we make explicit on a specimen. Considering that $12 = 2 \times 2 \times 3$, in order to arrive at the quotient of 417 by 12, we first determine the quotient of 417 by 2, which is the quotient of 416 by 2, i.e. 208. We then calculate the quotient of 208 by 2, which is simply 104; and finally we determine the quotient of 104 by 3, which is the same as the quotient of 102 by 3, or 34. The quotient of 417 divided by 12 is "therefore" 34. (Indeed, $417 = 34 \times 12 + 9$.) The inverted commas that surround *therefore* hint at the fact that many people—including mathematics teachers—will highly doubt the validity of this technique, on the grounds that it leads one to carelessly get rid of successive remainders. This paves the way for another key notion that ATD hinges on: the notion of *technology*. This word is used in ATD with its etymological value: as the suffix *-logy* indicates, a technology is a "discourse" on a given technique τ. This discourse is supposed, at least in the best-case scenario, both to *justify* the technique τ as a valid way of performing tasks t of type T and to throw light on the logic and workings of that technique, making it at least partially intelligible to the user. As concerns the technique of division shown above, it seems difficult to hit upon a full-fledged technology that would justify it, let alone *explain* it—if the technique is duly valid, *why* is it so? For lack of space, we shall leave these two mathematical tasks—justify and explain the aforementioned technique—to the perplexed reader.

A key point must be stressed. Owing to the presence of the suffix *-logy*, the word *technology* carries with it the idea of a *rational* discourse (about some *tekhne*—a Greek word meaning "a system or method of making or doing", that is, a technique or system of techniques). In the universe of ATD, there is no such thing as *universal* rationality. Every person x, every institution I, and every position p has its own rationality, afforded by the technologies present in its "praxeological equipment." Of course, persons and institutions strive to indulge their "rationality" or even to impose it upon others. The interplay between competing rationalities is a major aspect of what it is the mission of didactics to explore.

We have now arrived at a crossroads. It appears that no technological justification is self-sufficient: it relies on elements of knowledge of a higher level of generality, which, whenever they do not go unnoticed—they often do—, sound more abstract, more ethereal, oftentimes abstruse, as if they expressed the point of view of a far removed, pure spectator—a *theoros*. In ATD, such items of knowledge, sometimes dubbed "principles" (or "postulates", etc.), compose the *theory* that goes with the triple formed by the *type of tasks* T, *technique* τ, and *technology*. This theoretical component is denoted by the letter Θ ("big theta") while the technology is denoted by (small) θ. We thus arrive at a quadruple traditionally denoted by $[T / \tau / \theta / \Theta]$. It is this quadruple that we call *a* praxeology; it is called a *punctual* praxeology because it is organised around the type of tasks T, considered as a "point".

It should be clear that, by its very definition, ATD's notion of theory already subsumes case 3 of the English *Wiktionary*'s definition of *theory*: "The underlying principles or methods of a given technical skill, art etc., as opposed to its practice. [from 17th c.]." Let us take a step forward. A central tenet of ATD is that all "knowledge" can be modelled in terms of praxeologies. The "praxeological equipment" of a person x or institutional position p is defined to be the more or less integrated system of *all* the praxeologies that the person x or a person x' in position p can draw upon to do what this person is led to do. A praxeology can be denoted by the letter $\wp$ (called "Weierstrass p"). It can be construed as the union of two parts or "blocks": the *praxis* part $\Pi = [T / \tau]$, also called the *practico-technical* block, and the *logos* part $\Lambda = [\theta / \Theta]$ or *technologico-theoretical* block. One can write: $\wp = \Pi \oplus \Lambda = [T / \tau] \oplus [\theta / \Theta] = [T / \tau / \theta / \Theta]$. The operation $\oplus$ is sometimes called the *amalgamation* of the *praxis* and *logos* parts. The amalgamation of Π and Λ should be interpreted as a *dialectic* process of "sublation" [2] through which the *praxis* and *logos* parts are at the same time negated as isolated parts but preserved as partial elements in a synthesis, which is the praxeology $\wp$. Let us for a moment relabel "knowledge part" the *logos* part and "know-how part" the *praxis* part of a praxeology $\wp$. The dialectic sublation of "knowledge" and "know-how" that $\wp$ is supposed to achieve is hardly ever actualized. More often than not, the *praxis* and the *logos* observable in a person's or institutional position's praxeological equipment do not fit well together. The *praxis* block may be poorly developed while the *logos* part seems to be ahead of the game—a state of things often expressed by saying something like "he knows the theory, but can't apply it." Or the *praxis* part seems to be going smoothly but the *logos* part is so poor that it fails to substantially explain or justify the featured technique, which is consequently turned into a mere "recipe." The failure to arrive at a "well-balanced" praxeology is the rule, not the exception—a key phenomenon that we will now dwell upon.

INCOMPLETENESS AND IMPLICITNESS IN PRAXEOLOGIES

When it comes to discussing praxeological matters, people are prone to using metonymies or, more precisely, synecdoches [3]. This synecdochic bent generally selects as a derived name some (supposedly) "noble" part or feature of the thing to name. The widely shared propensity to metonymize shows up in particular in the use of the word *knowledge*—which is the "lofty" part of a praxeology—to name the *whole* praxeology. It is even more manifest in the generalized use of *theory* as including not only what ATD calls *technology*, but also the *praxis* part and, therefore, the whole praxeological matter. In common parlance, *theory* refers usually, though somewhat fuzzily, to a complex of praxeologies sharing a common "theory" (in a sense acknowledged by the naming institutions). Such a "body of knowledge" can be denoted by the formula $[T_{ij} / \tau_{ij} / \theta_i / \Theta]$ with $i = 1, ..., n$ and $j = 1, ..., m_i$, where the theory Θ "governs" all the technologies θ_i, each technology θ_i "governing" in turn the techniques τ_{ij}. Such a praxeology goes by the name of *global* praxeology. It is this generic analysis that ATD offers when one comes to speak of, for instance, "group theory" or "number theory" or "chaos theory" or "knot theory," etc. It is to be observed that, in doing so, the praxeological complex to which one refers is defined "in intension" rather than "in extension." It allows one to identify conceptually the *possible* content of the praxeological complex, while its real "extension" remains somewhat unspecified. Of course, it is risky to be so unmethodical when it comes to describing praxeological organisations. Naming a part to mean the whole leads to forget or neglect other parts. Therefore, the resulting praxeologies cannot be efficient tools for action—just as a car stripped down to the engine is of little avail to travel (even if, again metonymically, "motor" can be used to refer to the whole car).

This is however one aspect only of the problem of incompleteness in praxeologies. Any praxeology whatsoever can be said to be incomplete, be it technically, technologically or theoretically. And it is the fate of all praxeologies to continually go through a process which can further the development of any of their constituent parts: the technique can be further "technicized", the technology "technologized", and the theory "theorized". Consider the following easy example relating to the century-old "rule of three", that of the so-called "unitary method", which L. C. Pascoe in his *Arithmetic* (1971) introduces as "helpful to those who initially have difficulties with the ideas of ratios" (p. 64). Traditional arithmetical techniques were essentially *oral*: to do mathematics, one had to *say* something, in order to arrive at the sought-for result. For instance, if it is known that 132 tickets cost £165, how much will be paid for 183 tickets? The right "saying" goes somewhat as follows [4]: "If 132 tickets cost £165, then 1 ticket costs 132 times *less*, or £$\frac{165}{132}$; and 183 tickets cost 183 times *more*, or £$\frac{165}{132} \times 183$, that is £228.75." Here the type of tasks T is clearly delineated; and so is the propounded technique τ_0. As is often the case with arithmetic, the technology θ of τ is essentially embodied in the "technical discourse" above, that both activates τ and explains—makes plain—its logic, thereby justifying it. As always, the "justifying efficacy" of θ depends much on the apparent "naturalness" of the supposedly self-evident reasoning conveyed by the technical discourse recited (if n cost p, then 1 costs p/n, etc.). There exist, of course, other techniques. Some centuries ago, people would have said something like "132 is to 165 as 183 is to price p", writing down the "proportion" 132:165::183:p. Using the (technological) assertion that, in such a proportion, the product of the "means" (i.e. 165 and 183) equals the product of the "extremes" (i.e. 132 and p), they would have arrived at the equation $165 \times 183 = 132 \times p$, which gives $p = \frac{165 \times 183}{132}$. This formula appears to agree with the one found using τ_0, provided one knows the (technological) equality $\frac{a \times c}{b} = \frac{a}{b} \times c$. But this age-old technique τ_{-1} was technologically—not technically—*more demanding*, because the reason why the key technological assertion (about means and extremes) is true remains hidden—which, for most users, turns τ_{-1} into a recipe.

The technique τ_0 can be modified in (at least) two subtly different ways. One consists in introducing an easy technological notion from daily life, that of *unit price*, which leads to a technical variant of τ_0: "If 132 tickets cost £165, then 1 ticket costs 132 times *less*, or £$\frac{165}{132}$, that is £1.25; and m tickets will cost m times *more*, or £$1.25 \times m$." This technical variant τ_{01} is a little bit more complex technically (by contrast, τ_0 skips the calculation of the unit price, though the technological concept of unit price is already implicitly present); but it provides more *technological comfort* to the layman. Another variant results from a decisive *theoretical* change. While people generally understand the expression "number of times" as referring to a *whole* number of times, as was the case in the tickets problem, a major advance in the history of numbers consisted in regarding *fractions* as true numbers, on a par with what came to be called *natural* numbers—fractional numbers being called by contrast *artificial* numbers. A second step forward, not yet taken by so many people, consists in extending the scope of the expression "number of times" to include *fractional* numbers, so that, for instance, 183 is $\frac{183}{132}$ times 132 (i.e. $183 = 132 \times \frac{183}{132}$), from which it follows that the price of 183 tickets is $\frac{183}{132}$ times the price of 132 tickets, or $\frac{183}{132}$ times £165, that is £165 × $\frac{183}{132}$ (which is yet another resolvent). As long as one accepts to think in terms of *fractional* number of *times*, we have a new technique, τ_{02}, much more powerful and comfortable than τ_0 or τ_{01}. Knowing for instance that the price of 2988 tickets is £3735, we can now say that the price of 2012 tickets will be $\frac{2012}{2988}$ times the price of 2988 tickets, i.e. £3735 × $\frac{2012}{2988}$; etc. While the variation leading to τ_{01} only called for a rather easy modification in the technique's technological environment, here the change affects the *theory* itself, which in turn leads to a new technological concept, that of a fractional number of times.

In mathematics as well as the sciences, praxeologies turn out to be no less incomplete than in other fields of human activity. Many aspects of a praxeology's incompleteness are in fact linked to the impression of "naturalness" that so many people feel when they use (or even observe) this praxeology. Of course, the notion of naturalness undergoes institutional variations—let alone personal interpretations. But it is too often assumed that what is natural is, by definition, an unalterable given that does not have to be "justified." This, of course, runs contrary to the scientific tradition, of which it is the ambition to unveil the figments of institutional or personal imagination. Thus the French mathematician Henri Poincaré (1902, p. 74) regarded the principle of mathematical induction as "imposed upon us with such a force that we could not conceive of a contrary proposition." However, almost at the same time, progress in mathematics showed that this supposedly self-existent principle could be derived from the well-ordering principle [4]. The same phenomenon had happened more than two centuries earlier. The leading character was then John Wallis. According to Fauvel, Flood, and Wilson (2013), here is what happened:

> On the evening of 11 July 1663, he lectured in Oxford on Euclid's parallel postulate, and presented a seductive argument purporting to derive it from Euclid's other axioms. As Wallis observed, his argument assumes that similar figures can take different sizes. Wallis found this assumption very plausible, and if it were true then the parallel postulate would be a consequence of the other axioms of Euclid. It does, however, imply a remarkable result: in any geometry in which the parallel postulate does not hold, that similar figures would have to be identical in size as well as in shape, and so scale copies could never be made. (pp. 129–130)

Seventy years later, Girolamo Saccheri was to observe that Wallis "needed only to assume the existence of two triangles, whose angles were equal each to each and sides unequal" (Bonola, 1955, p. 29). Wallis's proof of the parallel postulate [5] opened the way to a major change that we can subsume under a broader historical pattern. By making *explicit* a theoretical property of Euclidean space—"To every figure there exists a similar figure of arbitrary magnitude" (Bonola, 1955, p. 15)—, Wallis reduced the incompleteness (in ATD's sense) of Euclidean geometry as a praxeological field. But he contributed much more to the mathematical sciences: he discovered a *constraint* that, until then, had been taken for granted (and thus ignored) and which turned out to be crucial in the development of geometry, in that it drew a clear demarcation line between Euclidean geometry and the yet to come non-Euclidean geometries.

At this point, we must introduce another key notion of ATD: that of *condition*, stealthily used in the behavioural sciences (through the idea of conditioning or being conditioned) and akin to more widespread notions such as *cause*, *variable*, and *factor*. Didactics is defined in ATD as the science of the conditions of diffusion of knowledge to persons and within institutions. More generally, ATD views any science—including mathematics—as studying a certain kind of conditions with a bearing on human life and its environments. In this respect, given an institutional position p, it is usual (and useful) to distinguish, among the set of conditions considered, those that could be *modified* by the people occupying position p, and those which cannot be altered by these people (though they could be modified by those in some position $p' \neq p$). Any science seeks to accrue knowledge and know-how in order to make the most of prevailing conditions and, in the case of constraints, to create new positions for which these constraints become modifiable conditions. Now, before doing so, it is necessary to *identify* such conditions and constraints, and this is precisely what happens in the Wallis episode, where the Euclidean constraint of invariance by similarity is brought out as a key theoretical property. At the same time, revealing some constraint usually brings forth alternative conditions that had gone unnoticed until then—non-Euclideanism, in the case at hand—and which become new objects of study. It must be stressed here that a science does not know in advance the complete set of conditions and constraints it has to cope with: constructing this set is, by nature, a never-ending task. All these considerations extend to any field of activity, whose praxeological equipments are the outcomes of facing sui generis conditions and constraints. We have now arrived at a position where it makes sense to revert to the question from which we started.

WHAT IS A THEORY?

It must be emphasized here that the interrelated notions of technique, technology and theory do not refer so much to "things" as to *functions*. A technique is a construct which, under appropriate conditions, performs a determined function—the technical function. The same may be said about technology and theory, which respectively perform the technological and theoretical functions. Up to a point, these last two functions look weakly distinguishable—indeed, any contrastive definition is sure to be plagued with counterexamples. Obviously, there are some general criteria allowing one to discern the technological from the theoretical: the first of them is regarded as more concrete, more specific and straightforward, while the second one is approached as being more abstract, more general, more meditative and far-fetched, as if it were reminiscent of its origins. In addition, as has been already highlighted, in an intellectual tradition that has persisted to this day, the second one is valued more highly than the other is. However, these considerations may impede the recognition of an essential phenomenon: the use which is often made of words like *theory* refers to the *explicit* aspects of an entity which we described as definitely subjected to inexplicitness and incompleteness.

From the point of view of ATD, it appears that the technological and theoretical components of a praxeological organisation—that is to say, its *logos* part—are almost always misidentified, because the usual view of them tends to focus on their "explicit" part, which looks generally pretentious and assumptive. This tendency clearly shows through the case 2 of the definition of *theory* given by the English *Wiktionary*: "(sciences) A coherent statement or set of ideas that explains observed facts or phenomena, or which sets out the laws and principles of something known or observed." This of course is representative of a dominant theory about... theories. Moreover, *theory* is often liberally used to label what boils down to a few guidelines or precepts which, taken together, do *not* function as the theory of any clearly identified object; for a theory should always be a theory *of something*, built around the scientific ambition to study this "something".

The metonymic use of *theory* is no problem in itself: when one says that ATD is a theory of "the didactic", *theory* refers, as is usual in mathematics for example, to the whole of a praxeological field. But it is a symptom of our propensity to give the word free rein with the uneasy consequence that the debate on theory is deprived of its object. By contrast, ATD conduces to focus the research effort on examining the implicit, unassuming or even wanting parts of technologies and theories. It then appears that a theory is made up of two main components, that we may call its "emerged part" and "immersed part". To avoid engaging here in a titanic work, we summarize in two points the constant lesson that praxeological analysis consistently teaches us. Firstly, the immersed part of a theory—in mathematics and, as far as we know, elsewhere—is replete with inexplicit tenets that are necessary to keep the emerged part afloat. Secondly, these tenets have surreptitious, far-reaching consequences, which often go unnoticed and usually unexplained at both the technological and the technical levels. What people do and how they do it owes much to "thoughts" unknown to them—unknown, not unknowable.

In ATD a theory is thus a hypothetical reality that assumes the form of a (necessarily fuzzy) set of explicit and implicit statements about the object of the theory. A theory is in truth the current state of a dialectic process of theorisation of which it offers an instantaneous and partial view that may prove delusive. The study and exploration of a theory is tantamount to furthering the very process of theorisation. One main feature of this process is that it allows for the expansion of too often ad hoc, punctual praxeologies $[T / \tau / \theta / \Theta]$ into deeply-rooted global praxeologies $[T_{ij} / \tau_{ij} / \theta_i / \Theta]$. The process of theorisation, as well as the networking of theorisations, has thus a liberating effect, in which, by the way, the use of well-chosen terms and symbolic notations helps achieve mental hygiene and theoretical clarity in bringing about what Bachelard once called the asceticism of abstract thought.

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ENDNOTES

1. See for instance the list proposed on the page at http://www.collinsdictionary.com/dictionary/english/theory.

2. The word *sublation* is the traditional rendering in English of Hegel's notion of *Aufhebung*. According to *Wikipedia* ("*Aufheben*", n.d.), "in sublation, a term or concept is both preserved and changed through its dialectical interplay with another term or concept. Sublation is the motor by which the dialectic functions."

3. A synecdoche is a phrase in which a part of something is used in order to refer to the whole of it.

4. See at http://en.wikipedia.org/wiki/Mathematical_induction#Equivalence_with_the_well-ordering_principle

5. For Wallis's proof in modern form, see, for example, Martin, 1975, pp. 273–274.